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THE BATALIN-VILKOVISKY METHOD OF QUANTIZATION MADE EASY

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Abstract

Odd time was introduced to formulate the Batalin-Vilkovisky method of quantization of gauge theories in a systematic manner. This approach is presented emphasizing the odd time canonical formalism beginning from an odd time Lagrangian. To let the beginners have access to the method essential notions of the gauge theories are briefly discussed, and each step is illustrated with examples. Moreover, the method of solving the master equation in an easy way for a class of gauge theories is reviewed. When this method is applicable some properties of the solutions can easily be extracted as shown in the related examples.

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I. INTRODUCTION

Theoretical aspects of the elementary particles are discovered in terms of gauge theories. The most common ones are the Yang-Mills type theories whose quantization is well understood. However, especially after the construction of the covariant string field theories (for a review see [1]), we learnt that there are some other interesting gauge theories which do not share the same properties with the Yang-Mills theory: there may be open gauge algebras, and/or the gauge generators may be linearly dependent (reducible gauge theory). In the latter case it is sometimes possible to choose a set of gauge transformations which behave like the gauge transformations of Yang-Mills theory. However, even if this choice is possible usually it destroys a manifest symmetry of the original system like Lorentz invariance, which one prefers to keep.

The most efficient method of quantizing reducible gauge theories whose gauge algebra is closed or open is given by Batalin and Vilkovisky (BV) [2]. They offered a systematic way of finding the full action which can be used in the related path integrals. Unfortunately, this method appears discouraging to the beginners because the machinery used to formulate the method is *ad hoc*: the reason of introducing antifields and antibrackets is obscure.

The essential step in the application of the BV method of quantization is to solve the (BV-) master equation. However, obtaining the desired solution is usually cumbersome. Moreover, the solutions are usually very complicated, so that extracting some algebraic or geometric properties of them is not easy.

There are some excellent reviews [3] and books [4] on this subject, in which one can find some different applications of the method as well as discussions of some general aspects of it like the structure equations resulting from the master equation. However, the *ad hoc* definitions of the method are kept, and there is no hint of solving the master equation for complicated systems in an easy way.

A solution to the former problem was given in terms of the "odd time" dynamics [5], and a general as well as an easy solution of the master equation for a vast class of gauge

theories is found [6], inspired by the odd time formalism of the BV method. However, a complete discussion of odd time Lagrangian and the canonical formulation resulting from it was missing. Moreover, neither the odd time dynamics nor the general solution were presented in a complete and pedagogical manner.

The aim of this article is to present the BV method of quantization without refering to any *ad hoc* definition and to give an application of it to a general class of gauge theories, in a way which renders easy the access to the method and its applications.

In Section II first the basic concepts of gauge theories are presented in terms of some examples. The examples chosen are appropriate to illustrate close and open gauge algebras and linearly dependent (reducible) gauge generators. Then, the reducibility conditions suitable for applying the BV method of quantization are given. Ghost and ghost of ghost fields are introduced in terms of path integrals, and the Becchi-Roulet-Stora-Tyutin (BRST) symmetry [7] for an irreducible system is discussed.

We devote Section III to the BV method of quantization. Odd time approach is presented by discussing an odd time Lagrangian and the related canonical formalism in detail. Then, a general solution of the master equation which embraces a vast class of gauge theories is discussed and applied to the examples given in Section II. Some properties of the proper solutions of the examples are also presented.

II. SOME FEATURES OF GAUGE THEORIES

A. Generalities

Here we briefly recall some properties of gauge theories illustrated by examples.

Let us deal with a theory given by

$$\mathcal{A}[\phi] = \int d^d x \mathcal{L}(\phi, \partial\phi/\partial x), \tag{2.1}$$

where the Grassmann parity of the fields are $\epsilon(\phi^i) = 0$ (commuting) or 1 (anticommuting) mod 2, and $i = 1, \cdots, n$. It is supposed that the action possesses at least one stationary point ϕ_0^i :

$$\frac{\delta \mathcal{A}}{\delta \phi^i}|_{\phi_0^i} = 0.$$

For the sake of simplicity let us deal with bosonic ϕ^i. When the fields ϕ^i are transformed by some infinitesimal local fields $\alpha^a(x)$; $a = 1, \cdots, m$,

$$\delta_\alpha \phi^i = R_a^i(\phi)\alpha^a(x), \tag{2.2}$$

if the action remains invariant

$$\delta_\alpha \mathcal{A} = \int d^d x \frac{\delta \mathcal{A}}{\delta \phi^i} R_a^i \alpha^a = 0, \tag{2.3}$$

up to surface terms (or $\delta_\alpha \exp\mathcal{A} = 0$), the action (2.1) defines a gauge theory. R_a^i and α^a are gauge generators and gauge parameters.

We assume that all of the gauge transformations can be generated by R_a^i, so that the commutator of two gauge transformations can be written in terms of the generators R_a^i, up to terms vanishing on mass shell:

$$[\delta_\alpha, \delta_\beta]\phi^i \equiv \frac{\partial R_{[a}^i}{\partial \phi^j} R_{b]}^j \alpha^a \beta^b = F_{ab}^c(\phi) R_c^i \alpha^a \beta^b + \frac{\delta \mathcal{A}}{\delta \phi^j} K_{ab}^{ji} \alpha^a \beta^b. \tag{2.4}$$

Here, [ ] denotes antisymmetrization in the indices which are within them. If K vanishes identically, gauge transformations form an algebra, and moreover if F is independent of ϕ

it is a Lie algebra. In the case where K does not vanish, gauge transformations still satisfy an algebra on mass shell, hence it is called an open gauge algebra.

The generators R^i_a, enumerated by a can be linearly independent or dependent. In the former case the theory is named irreducible, and in the latter case reducible theory. i.e. if there exists some (non-zero) gauge parameters $\alpha^a_{(r)}$ satisfying

$$R^i_a \alpha^a_{(r)}|_{\phi_{(0)}} = 0,$$

the gauge theory is reducible.

Before discussing the conditions of reducibility in general, which are adequate to use the BV method of quantization, let us give some examples to illustrate the cases discussed above.

1. Examples

a. Yang-Mills Theory: It is defined in terms of the second order action

$$L_0 = \frac{-1}{4} \int d^4x \; F^a_{\mu\nu} F^{\mu\nu}_a, \tag{2.5}$$

where in the differential form notation $F = d \wedge A + A \wedge A$. Gauge fields are in the adjoint representation of the gauge group $SU(N)$, $A_\mu \equiv A^a_\mu t_a$, where t_a are the generators of the Lie algebra:

$$[t_a, t_b] = f_{ab}{}^c t_c.$$

(2.5) is invariant under the gauge transformations

$$\delta A_\mu = D_\mu \alpha \; , \tag{2.6}$$

where $D = d + [A, \;]$ is the covariant derivative. As one can easily observe the theory is irreducible, i.e. D does not possess any non-trivial zero eigenvalue vector:

$$D^\mu \beta_k = 0 \Longrightarrow \beta_k = 0, \tag{2.7}$$

and the gauge transformations satisfy the Lie algebra

$$[\delta_\alpha, \delta_\rho] A^a_\mu = f_{bc}{}^a D_\mu \alpha^b \rho^c.$$

b. The Self-interacting Antisymmetric Tensor Field: The action [8] (we suppress Tr which is over the group indices, and define Tr $t_a t_b = \delta_{ab}$)

$$L_0 = -\int d^4x \; [B_{\mu\nu}(d \wedge A + A \wedge A)^{\mu\nu} - \frac{1}{2} A_\mu A^\mu], \tag{2.8}$$

is invariant under the transformations

$$\delta_\Lambda B_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} D^\rho \Lambda^\sigma, \;\; \delta_\Lambda A_\mu = 0.$$

Obviously, gauge algebra closes off shell

$$[\delta_\Lambda, \delta_\Sigma](B_{\mu\nu}, A_\mu) = 0. \tag{2.9}$$

However, for $\Lambda_\mu = D_\mu \alpha$, the gauge transformation vanishes on shell $\delta_\Lambda B|_{F=0} = 0$. In other terms the gauge generators

$$R^\sigma_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} D^\rho, \tag{2.10}$$

possess non-trivial zero eigenvalue vectors D_σ on mass shell:

$$R^\sigma_{\mu\nu} D_\sigma|_{F=0} = 0. \tag{2.11}$$

Hence, this a reducible theory. Moreover, D does not possess non-trivial zero eigenvalue vector (2.7).

c. Chern-Simons theory in $d = 2p + 1$: For $p = 1,\ 2,\ 3 \cdots$, it is given in terms of the action

$$L_d = \frac{1}{2} \int_{M_d} \Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \Big). \tag{2.12}$$

If the gauge field is defined as

$$A = \phi + \psi \equiv \sum_{i=0}^{p-1} \phi_{2i+1} + \sum_{i=0}^{p} \psi_{2i}, \tag{2.13}$$

where ϕ_{2i+1} and ψ_{2i} are Lie-algebra valued, respectively, bosonic $2i+1$-form and fermionic $2i$-form, the Chern-Simons action (2.12) yields

$$L_d = \frac{1}{2}\int_{M_d}\left(\phi\wedge d\phi + \frac{2}{3}\phi\wedge\phi\wedge\phi + \psi\wedge D_\phi\psi\right), \tag{2.14}$$

where $D_\phi \equiv d + [\phi,\ ]$. (2.14) is followed from the fact that in the integral only the terms possessing odd differential form degree survive. This theory is introduced in ref. [9].

The action (2.14) is invariant under the gauge transformations

$$\delta_\Sigma A = d\Sigma + [A,\Sigma] \equiv \begin{pmatrix} D_\phi & \psi \\ \psi & D_\phi \end{pmatrix}\begin{pmatrix} \Lambda \\ \Xi \end{pmatrix}, \tag{2.15}$$

where the gauge parameter is

$$\Sigma = \Lambda + \Xi \equiv \sum_{i=0}^{p-1}\Lambda_{2i} + \sum_{i=0}^{p-1}\Xi_{2i+1}. \tag{2.16}$$

Λ and Ξ are bosonic and fermionic, respectively. For some values of Σ the gauge transformations (2.15) vanish on mass shell. Indeed, when the equations of motion

$$d\phi + [\phi,\phi] - \psi^2 = 0,\ \ D_\phi\psi = 0, \tag{2.17}$$

are satisfied, the gauge generators generating (2.15) are linearly dependent. Moreover the zero eigenvalue vectors are also linearly dependent:

$$Z_m Z_{m+1} = 0,\ \ m = 0,\cdots 2p-2, \tag{2.18}$$

where Z_0 is the gauge generator of (2.15) and

$$Z_{2m} = \begin{pmatrix} D_\phi & \psi \\ \psi & D_\phi \end{pmatrix}, Z_{2m+1} = \begin{pmatrix} D_\phi & -\psi \\ -\psi & D_\phi \end{pmatrix}.$$

Observe the difference between the reducibility of this theory and the previous one.

d. The Gauge Theory of Quadratic Lie Algebras: The algebra generated by T_a

$$[T_a, T_b] = f_{ab}{}^{c}T_c + V_{ab}^{cd}T_cT_d + k_{ab}, \tag{2.19}$$

is known as quadratic Lie algebra, even if it is not a Lie algebra. It is obtained by deforming the Lie algebra given by the structure constants ${f_{ab}}^c$. The constants f, V, and k possess the symmetry properties

$$ {f_{ab}}^c = -{f_{ba}}^c,\ V_{ab}^{cd} = -V_{ba}^{cd},\ V_{ab}^{cd} = V_{ab}^{dc},\ k_{ab} = -k_{ba}. \tag{2.20} $$

Moreover, they should be chosen to obey

$$ \begin{aligned} {f_{[ab}}^d {f_{c]d}}^e &= 0, \\ {f_{[ab}}^d V_{c]d}^{ef} + V_{[ab}^{df} {f_{c]d}}^e + V_{[ab}^{ed} {f_{c]d}}^f &= 0, \\ V_{[ab}^{de} V_{c]d}^{fg} &= 0, \\ {f_{[ab}}^d k_{c]d} &= 0, \\ V_{[ab}^{de} k_{c]d} &= 0, \end{aligned} \tag{2.21} $$

because of the Jacobi identities.

Gauge theory of this algebra in 2–d space-time is given by the Lagrange density [10]

$$ \mathcal{L} = -\frac{1}{2}\epsilon^{\mu\nu}\{\Phi_a(\partial_\mu h_\nu^a - \partial_\nu h_\mu^a + {f_{bc}}^a h_\mu^b h_\nu^c + V_{bc}^{ad}\Phi_d h_\mu^b h_\nu^c) + k_{ab} h_\mu^a h_\nu^b\}, \tag{2.22} $$

which leads to the equations of motion

$$ \epsilon^{\mu\nu}(D_\nu \Phi_a + k_{ab} h_\nu^b) = 0, \tag{2.23} $$

$$ \epsilon^{\mu\nu}(\partial_\mu h_\nu^a - \partial_\nu h_\mu^a + {f_{bc}}^a h_\mu^b h_\nu^c + 2V_{bc}^{ad}\Phi_d h_\mu^b h_\nu^c) = 0. \tag{2.24} $$

We used the definition

$$ D_\mu \Phi_a \equiv \partial_\mu \Phi_a + \Phi_c f_{ab}^c h_\mu^b + \Phi_c \Phi_d V_{ab}^{cd} h_\mu^b. $$

The action (2.22) is invariant under the gauge transformations

$$ \delta h_\mu^a = \partial_\mu \lambda^a + {f_{bc}}^a h_\mu^b \lambda^c + 2V_{bc}^{ad}\Phi_d h_\mu^b \lambda^c, \tag{2.25} $$

$$ \delta \Phi_a = {f_{ba}}^c \Phi_c \lambda^b + V_{ba}^{cd}\Phi_c\Phi_d\lambda^b + k_{ab}\lambda^b, \tag{2.26} $$

which satisfy

$$[\delta_\lambda, \delta_\eta] h^a_\mu = \delta_\kappa h^a_\mu - 2\lambda^c \eta^d V^{ab}_{cd}(D_\mu \Phi_b + k_{be} h^e_\mu) \tag{2.27}$$

$$[\delta_\lambda, \delta_\eta] \Phi_a = \delta_\kappa \Phi_a, \tag{2.28}$$

where

$$\kappa^a \equiv ({f_{bc}}^a + 2V^{ad}_{bc}\Phi_d)\lambda^b \eta^c.$$

Although, the commutator (2.27) leads to an algebra only on mass shell, the gauge generators of (2.25)-(2.26) are linearly independent. Hence this theory is an example to an irreducible gauge theory whose gauge generators satisfy an open algebra.

B. Reducibility Conditions and Ghost Fields

1. Irreducible Gauge Theories

When we deal with the partition function† of a gauge theory

$$Z = \int [\mathcal{D}\phi^i] exp \int d^d x \mathcal{L},$$

the measure $[\mathcal{D}\phi^i]$, should take into consideration that due to gauge invariance some of the integrals over fields are irrelevant and lead to infinities. Eliminating these irrelevant degrees of freedom usually causes destruction of some manifest symmetries like covariance. Hence, one usually prefers to keep all of the original fields, but put some gauge fixing conditions. When the gauge generators R_a are linearly independent (irreducible), gauge fixing can be achieved in terms of the conditions

$$\chi_a(\phi) = 0,$$

†We can equivalently consider the Green's functions generating functional

$$Z[J] = \int [\mathcal{D}\phi^i] exp \int d^d x [\mathcal{L} + J\phi],$$

in terms of the gauge invariant sources J_i.

whose Grassmann parity is denoted as

$$\epsilon(\chi_a) = \epsilon_a.$$

Then the correct measure (or Haar measure) is

$$[\mathcal{D}\phi^i] = \prod_{i,x} d\phi^i(x)\ \delta(\chi_a(\phi))\mathrm{det}^{(-)^{\epsilon_a}} \left[\frac{\partial\chi_a}{\partial\phi^i}\frac{\partial(\delta\phi^i)}{\partial\alpha^b}\right],$$

where α is the gauge parameter. Let us define an effective action by putting the terms which are in the measure into the exponent. To achive this let us introduce the fields

$$\lambda^a,\ \eta^a,\ \bar{\eta}^a;\ \epsilon(\lambda^a) = \epsilon_a\ ;\ \epsilon(\eta^a) = \epsilon(\bar{\eta}^a) = \epsilon_a + 1\ .$$

Now, in terms of them we can write the related path integral as

$$Z = \int \prod_{i,x,a} d\phi^i(x) d\eta^a(x) d\bar{\eta}^a(x) d\lambda^a(x) \mathrm{e}^{-\mathcal{A}_{\mathrm{eff}}}, \tag{2.29}$$

where

$$\mathcal{A}_{eff} = \int d^dx\{\mathcal{L} + \lambda^a\chi_a + \bar{\eta}^a \left[\frac{\partial\chi_a}{\partial\phi^i}\frac{\partial(\delta\phi^i)}{\partial\alpha^b}\right]\eta^b\}. \tag{2.30}$$

Obviously λ^a are Lagrange multipliers and $\eta^a,\ \bar{\eta}^a$ are the so called ghost, antighost fields. Observe that λ_a possess the same statistics but the ghosts η^a and the antighosts $\bar{\eta}^a$ possess the opposite statistics of χ_a. Here, the ghosts η^a are introduced as some auxiliary fields. Hence, one should differ them from the original ones. To this aim, introduce the ghost number $\mathrm{N_{gh}}$, which is zero for the original fields ϕ^i and the Lagrange multipliers λ^a, but

$$\mathrm{N_{gh}}(\eta^a) = -\mathrm{N_{gh}}(\bar{\eta}^a) = 1.$$

2. Reducible Gauge Theories

As it is obvious from the above discussion, for a covariant quantization some ghost fields are needed. Batalin and Vilkovisky gave a way of performing this for theories which satisfy some conditions [2]:

a. First Stage Reducible Gauge Theories: Let us suppose that

$$R^i_{a_0} Z_1^{a_0}{}_{a_1}|_{\phi_0} = 0; \tag{2.31}$$

$a_0 = 1, \cdots, m_0;\ a_1 = 1, \cdots, m_1 < m_0$, are satisfied by some non-trivial (nonzero) Z_1, but Z_{1a_1} are linearly independent. Moreover, if

$$\begin{aligned} \text{rank } R^i_{a_0} &= m_0 - m_1 < n; \\ \text{rank } Z_1^{a_0}{}_{a_1} &= m_1; \\ \text{rank } \frac{\delta^2 \mathcal{A}}{\delta\phi^i \delta\phi^j}|_{\phi_0} &= n - (m_0 - m_1), \end{aligned}$$

are satisfied, the theory is a first stage reducible gauge theory.

Grassmann parities are denoted as

$$\epsilon(R^i_{a_0}) = \epsilon_i + \epsilon_{a_0};\ \epsilon(Z_1^{a_0}{}_{a_1}) = \epsilon_{a_0} + \epsilon_{a_1}.$$

Because of the linear dependence of R_a, not all of the original gauge transformations are relevant. To find an effective action for covariant quantization we introduce the zero stage ghost fields $\eta_0^{a_0}$, whose Grassmann parity is $\epsilon_{a_0} + 1$, and ghost number 1, to write, similar to (2.30),

$$\mathcal{A}^0_{eff} = \int d^d x \{\mathcal{L} + \lambda^{a_0}\chi_{a_0} + \bar{\eta}_0^{a_0} \left[\frac{\partial \chi_{a_0}}{\partial \phi^i} R^i_{b_0}\right] \eta_0^{b_0}\}. \tag{2.32}$$

However, the transformations

$$\delta\eta_0^{a_0} = Z_1^{a_0}{}_{a_1}\alpha_1^{a_1},$$

leave (2.32) invariant. Now the ghost field η_0 behaves like a gauge field. So that, we introduce some other ghosts (ghosts of ghosts), and antighosts

$$\eta_1^{a_1}, \bar{\eta}_1^{a_1};\ \epsilon(\eta_1^{a_1}) = \epsilon(\bar{\eta}_1^{a_1}) = \epsilon_{a_1} + 1;\ \mathrm{N_{gh}}(\eta_1^{a_1}) = -\mathrm{N_{gh}}(\bar{\eta}_1^{a_1}) = 2,$$

Lagrange multipliers and gauge fixing conditions, respectively,

$$\lambda_1^{a_1},\ \chi^1_{a_1};\ \epsilon(\lambda_1^{a_1}) = \epsilon(\chi^1_{a_1}) = \epsilon_{a_1};\ \mathrm{N_{gh}}(\lambda_1^{a_1}) = \mathrm{N_{gh}}(\chi^1_{a_1}) = 0.$$

We may choose the gauge fixing conditions $\chi^1_{a_1}$ depending only on η_0, so that the partition function

$$Z' = \int \prod_{i,x,a_0,a_1} d\phi^i(x) d\eta_0^{a_0}(x) d\bar{\eta}_0^{a_0}(x) d\lambda_0^{a_0}(x) d\eta_1^{a_1}(x) d\bar{\eta}_1^{a_1}(x) d\lambda_1^{a_1}(x) \mathrm{e}^{-\mathcal{A}'_{\mathrm{eff}}}, \tag{2.33}$$

is written in terms of the effective action

$$\mathcal{A}'_{eff} = \mathcal{A}^0_{eff} + \int d^dx \{\lambda_1^{a_1}\chi^1_{a_1} + \bar{\eta}_1^{a_1}\left[\frac{\partial \chi^1_{a_1}}{\partial \eta_0^{a_0}} Z_1^{a_0}{}_{b_1}\right]\eta_1^{b_1}\}. \tag{2.34}$$

Therefore, the number of ghost fields depends on the level of reducibility.

b. l$^{\text{th}}$ *Stage Reducible Gauge Theories:* Suppose that in addition to (2.31)

$$Z_r^{a_{r-1}}{}_{a_r} Z_{r+1}^{a_r}{}_{a_{r+1}}|_{\phi_0} = 0,$$

$a_r = 1, \cdots, m_r$, are satisfied with non-trivial Z_r for $r = 1, \cdots, l$, and $Z_{l+1} = 0$. If they also satisfy

$$\text{rank } R^i_{a_0} = \beta_0 < n; \ \beta_0 = \textstyle\sum_{i=0}^l (-1)^i m_i$$

$$\text{rank } Z_r^{a_{r-1}}{}_{a_r} = \beta_r; \beta_r = \textstyle\sum_{i=r}^l (-1)^i m_i; \ r = 1, \cdots, l$$

$$\text{rank } \frac{\delta^2 \mathcal{A}}{\delta\phi^i \delta\phi^j}|_{\phi_0} = n - \beta_0,$$

the theory is a l$^{\text{th}}$ stage reducible gauge theory.

The Grassmann parities are denoted as

$$\epsilon(Z_r^{a_{r-1}}{}_{a_r}) = \epsilon_{a_{r-1}} + \epsilon_{a_r}.$$

Similar to the previous case we need to introduce ghost and ghost of ghost fields:

$$\eta_r^{a_r}; \ \epsilon(\eta_r^{a_r}) = \epsilon_{a_r}; \ \mathrm{N_{gh}}(\eta_r^{a_r}) = r + 1, \ r = 0, \cdots, l - 1.$$

One can enlarge the set of fields by introducing the related antighosts and Lagrange multipliers in an obvious manner to discuss the effective action similar to the previous cases.

To have an insight of the statistics of the ghost fields let us assume that the original fields ϕ_i, the gauge generators R, and all of the Z_r are bosonic. Then, the zero stage ghosts η_0 are anticommuting, the first stage ghosts η_1 are commuting and the rest is continued by alternating Grassmann parity.

C. Physical State Conditions in terms of the BRST Formalism

Effective actions introduced contain some irrelevant gauge fields, moreover some ghosts and auxiliary fields which are not physical. These fields should be isolated from the rest. A way of performing this is the use of BRST symmetry. Before explaining the method in general, we discuss this symmetry for the simplest case.

Let us deal with a bosonic, irreducible gauge theory whose gauge algebra closes off shell. Moreover, we suppose that the gauge fixing functions χ_a are linear in the fields ϕ_i, so that the effective action which can be used in path integrals (2.30), becomes

$$\mathcal{A}_{eff} = \int d^dx[\mathcal{L} + \lambda^a\chi_a + \bar{\eta}^a \left[\mathcal{O}_{ai}(x)R^i_b\right]\eta^b], \tag{2.35}$$

where $\mathcal{O}_{ai}(x)$ are some operators. This action is invariant under the transformation

$$\delta_Q\phi^i = R^i_a\eta^a,\ \delta_Q\eta^a = -\frac{1}{2}F^a_{bc}(\phi)\eta^b\eta^c,\ \delta_Q\bar{\eta}^a = -\lambda^a,\ \delta_Q\lambda^a = 0, \tag{2.36}$$

which is reminiscent of the gauge transformations (2.2)-(2.4) and defined such that

$$\delta^2_Q(\phi,\eta,\bar{\eta},\lambda) = 0. \tag{2.37}$$

δ_Q is the well known BRST transformation [7]. As an example let us deal with Yang-Mills theory. In the covariant gauge,

$$\partial^\mu A^a_\mu = 0,$$

the effective action reads

$$\mathcal{A}^{YM}_{eff} = \int d^4x[-\frac{1}{2}F^2_{\mu\nu} + \lambda\partial^\mu A_\mu + \bar{\eta}\partial^\mu D_\mu\eta]. \tag{2.38}$$

Obviously, η and $\bar{\eta}$ are fermionic fields. One can observe that (2.38) possesses the symmetry defined by

$$\delta_Q A_\mu = D_\mu\eta\ ;\ \delta_Q\eta^a = -\frac{1}{2}f^a_{bc}\eta^b\eta^c\ ;\ \delta_Q\bar{\eta} = -\lambda\ ;\ \delta_Q\lambda = 0. \tag{2.39}$$

Moreover, this transformation is nilpotent

$$\delta_Q^2(A, \lambda, \eta, \bar{\eta}) = 0.$$

$\delta_Q A_\mu$ can be obtained from the gauge transformation (2.6) by the replacement $\alpha \to \eta$.

A similar treatment of reducible theories is also available. Hence, let us deal with a gauge theory given in terms of an effective Lagrangian $\mathcal{L}_{\rm eff}(\Phi)$ and a nilpotent transformation δ_Q:

$$\delta_Q \mathcal{L}_{\rm eff}(\Phi) = 0,\ \ \delta_Q^2 \Phi_A = 0, \tag{2.40}$$

where Φ_A denote the needed ghosts, antighosts, Lagrange multipliers and the original fields.

By introducing the canonical momenta

$$P_A = \frac{\partial \mathcal{L}_{\rm eff}}{\partial (d\Phi_A/dt)},$$

one can write the BRST transformations in the phase space:

$$\delta_Q \Phi_A = \Omega_A(\Phi, P),$$

which leads to the Noether charge

$$Q = \int d^{d-1}x[\Omega_A(\Phi, P)P_A - K], \tag{2.41}$$

where K is defined to satisfy

$$\frac{\partial K}{\partial P_A} - \frac{\partial \Omega_A}{\partial P_A} P_B = 0. \tag{2.42}$$

By applying the usual canonical quantization procedure one can find a nilpotent operator

$$Q_{op}^2 = 0,$$

resembling the charge (2.41), if there does not exist any ordering anomaly. Then, one defines the physical states as

$$Q_{op}\psi_{phys.} = 0,$$

which are also defined to be on mass shell. In terms of the perturbative analysis one can show that existence of the charge Q_{op} satisfying the above mentioned properties is sufficient to eliminate the unphysical degrees of freedom [11].

Although, a charge which is related to Q_{op} which can be used to obtain the physical states is available even before gauge fixing in terms of the Batalin-Fradkin-Vilkovisky quantization scheme [12], it is out of the scope of this paper.

III. THE BATALIN-VILKOVISKY METHOD OF QUANTIZATION

A. Odd Time Formulation

Inspired by supersymmetry, one can introduce a superpartner of time, which we call "odd time" τ, satisfying

$$\tau^2 = 0.$$

Let us consider odd time dynamics in terms of the variable $q_\mu(x,\tau)$, where x indicates the usual time in particle case, and all of the coordinates in field theory. We use the same notation for the functions and the functionals. Moreover, the integrals over x are mostly suppressed. In contrary to the usual mechanics, an implicit odd time dependence does not make sense. One can always write

$$q_\mu(x,\tau) = q_\mu(x,0) + \tau q'_\mu(x,\tau),$$

where odd time derivative of q_μ is independent of τ. To emphasize this property we use the notation

$$q'_\mu(x,\tau) \equiv q'_\mu(x).$$

Hence, if we would like to describe the system in terms of an odd time Lagrangian L_o, it will be in the following form

$$L_o(q(x,0), q'(x), \tau) = L_o(q(x,0), q'(x), 0) + \tau L'_o(q(x,0), q'(x), \tau). \tag{3.1}$$

Obviously, L'_o is independent of τ.

Similar to the usual case one can define "odd time canonical momenta" as

$$p^\mu(x,\tau) = \frac{\partial L_o(q(x), q'(x), \tau)}{\partial q'_\mu(x)}. \tag{3.2}$$

L_o is supposed to be bosonic, so that p^μ possesses the opposite statistics of q_μ. Hence, we deal with a supermanifold possessing an equal number of fermionic and bosonic coordinates.

On such a manifold there exist an even as well as an odd canonical two form. We would like to deal with the latter one [13].

Only the first derivative of a function with respect to odd time can be non-vanishing. Thus, for a canonical formalism it is sufficient to discuss only first order Lagrangians. In general one can deal with the two sets of variables

$$q_\mu \equiv (a_i(x,\tau), b_i(x,\tau)) = (a_i(x,0) + a'(x)\tau, b_i(x,0) + \tau b_i'(x)).$$

We choose odd time Lagrangian to be

$$L_o = a_i(x,0)b_i'(x) + a_i'(x)b_i(x,0) + a_i'(x)\tau b_i'(x) - S(a_i(x,0), b_i(x,0)). \tag{3.3}$$

Grassmann parity of the variables should be $\epsilon(a_i) = \epsilon(b_i) + 1$. Odd time canonical momenta are defined as

$$p_a^i = \frac{\partial_l L_o}{\partial a_i'} = b_i(x,\tau), \tag{3.4}$$

$$p_b^i = \frac{\partial_r L_o}{\partial b_i'} = a_i(x,\tau), \tag{3.5}$$

where right and left derivatives are related

$$\frac{\partial_r f(z)}{\partial z} = (-1)^{\epsilon(z)[\epsilon(f)+1]} \frac{\partial_l f(z)}{\partial z}.$$

L_o and right and left derivatives in the definitions of odd canonical momenta (3.4)-(3.5), are chosen to avoid (-1) factors.

We define the related odd Hamiltonian as

$$H_o \equiv a_i' p_a^i + p_b^i b_i' - a_i' \tau b_i' - L_o, \tag{3.6}$$

$$= S(a_i(x,0), b_i(x,0)). \tag{3.7}$$

In terms of this definition odd time independence of H_o is guaranteed:

$$\frac{\partial H_o}{\partial \tau} = 0. \tag{3.8}$$

Now, one can define an "odd Poisson bracket"(antibracket) in terms of a, b and their canonical momenta. Because of the constraints (3.4), (3.5) one can eliminate $p_a,\ p_b$ such that the basic odd Poisson brackets are

$$(a_i, b_j) = \delta_{ij}. \tag{3.9}$$

i.e. Observables $f,\ g$ are functions of $a_i,\ b_i$ and their odd Poisson bracket is

$$(f, g) = \frac{\partial_r f}{\partial b_i}\frac{\partial_l g}{\partial a_i} - \frac{\partial_r f}{\partial a_i}\frac{\partial_l g}{\partial b_i}. \tag{3.10}$$

The odd Poisson bracket (antibracket) has the following properties

$$\epsilon[(f, g)] = \epsilon(f) + \epsilon(g) + 1, \tag{3.11}$$

$$(g, f) = -(-1)^{[\epsilon(f)+1][\epsilon(g)+1]}(f, g), \tag{3.12}$$

$$\begin{aligned}(-1)^{[\epsilon(f)+1][\epsilon(g)+1]}(g, (l, f)) + (-1)^{[\epsilon(g)+1][\epsilon(l)+1]}(l, (f, g))& \\ + (-1)^{[\epsilon(l)+1][\epsilon(f)+1]}(f, (g, l)) = 0.&\end{aligned} \tag{3.13}$$

We should clarify the meaning of the derivative with respect to $q_\mu(x,\tau)$. It is demanded to satisfy

$$\frac{\partial}{\partial q_\mu} q_\nu - q_\nu \frac{\partial}{\partial q_\mu} = \delta^\mu_\nu. \tag{3.14}$$

The choice

$$\frac{\partial}{\partial q_\mu(x,\tau)} = \frac{\partial}{\partial q_\mu(x,0)} + \tau \frac{\partial}{\partial q'_\mu(x)}, \tag{3.15}$$

can be seen to satisfy (3.14) in the space of functions which are polynomials in q_μ.

Similar to the usual case let the odd time evolution of an observable f is generated by the odd time Hamiltonian S in terms of the odd Poisson bracket

$$f'(a, b) \equiv (S, f) = \frac{\partial_r S}{\partial b_i(x,0)}\frac{\partial_l f}{\partial a_i(x,\tau)} - \frac{\partial_r S}{\partial a_i(x,0)}\frac{\partial_l f}{\partial b_i(x,\tau)}, \tag{3.16}$$

where we used (3.15).

Equations of motion of the canonical variables are

$$a_i' \;= (a_i, S) \;= -\frac{\partial_l S(a(x,0), b(x,0))}{\partial b_i(x,0)}, \tag{3.17}$$

$$b_i' = (b_i, S) = \frac{\partial_l S(a(x,0), b(x,0))}{\partial a_i(x,0)}. \tag{3.18}$$

Observe that these agree with odd time equations of motion resulting from the odd time Lagrangian (3.3), if they are defined as

$$\frac{\partial_l L_o}{\partial a_i(x,\tau)} - \tau \frac{\partial_l L_o}{\partial a_i'(x)} = 0, \tag{3.19}$$

$$\frac{\partial_r L_o}{\partial b_i(x,\tau)} - \tau \frac{\partial_r L_o}{\partial a_i'(x)} = 0. \tag{3.20}$$

In (3.20) agreement is up to a sign factor if b is bosonic.

Moreover, $S(a(x,0), b(x,0))$ should be invariant under the odd time evolution (3.16):

$$(S,S) = \frac{\partial_r S}{\partial a_i(x,0)} \frac{\partial_l S}{\partial b_i(x,0)} - \frac{\partial_r S}{\partial b_i(x,0)} \frac{\partial_l S}{\partial a_i(x,0)} = 0, \tag{3.21}$$

which is known as (BV) master equation. By making use of (3.13) and (3.21) one can show that the second derivative of an observable f with respect to the odd time is vanishing

$$\frac{\partial^2 f}{\partial \tau^2} = (S,(S,f)) = 0.$$

In contrary to the usual case, invariance of Hamiltonian under the odd time evolution (3.21) is not trivially satisfied.

To attribute a physical content to the odd time dynamics one should specify the fields a, b and meaning of the odd time evolution (3.16). Here we use this formalism to formulate the BV method of quantization of gauge theories.

Let us deal with a gauge theory given in terms of an action $\mathcal{A}(\phi)$ invariant under the gauge transformations (2.2). Then, we identify the derivative with respect to the odd time with the BRST transformation (or charge):

$$\frac{\partial}{\partial \tau} \equiv \delta_{BRST}. \tag{3.22}$$

Now, analyse the reducibility of the gauge generators R^i_a and introduce the needed ghost fields possessing positive ghost number, as outlined in Section II B. Hence we assign

$$N_{\rm gh}(\frac{\partial}{\partial\tau}) = 1. \tag{3.23}$$

If the odd time Lagrangian (or Hamiltonian) (3.1) is physical, it has to satisfy

$$N_{\rm gh}(L_o) = 0. \tag{3.24}$$

Thus, to write an odd time Lagrangian which depends on ghost fields one should introduce some other fields possessing negative ghost number. Now, if q_i denote the original and the ghost fields, and p_i the odd time canonical momenta defined as (3.2), they should satisfy

$$N_{\rm gh}(p_i) = N_{\rm gh}(L_o) - (N_{\rm gh}(q_i) + 1), \tag{3.25}$$

which leads to

$$N_{\rm gh}(p_i) + N_{\rm gh}(q_i) = -1. \tag{3.26}$$

Therefore, the number of positive and negative ghost number components of the fields $a,\ b$ used to write L_o should be the same. To simplify the notation as well as to connect it to the usual one, let us rename the odd time independent components of a and b :

$$a_i(x,0) \equiv \Phi_i(x);\ \ b_i(x,o) \equiv \Phi^\star_i(x),$$

such that

$$N_{\rm gh}(\Phi_i) \geq 0,\ \ N_{\rm gh}(\Phi^\star_i) < 0.$$

Moreover, they should satisfy (3.26), namely

$$N_{\rm gh}(\Phi_i) + N_{\rm gh}(\Phi^\star_i) = -1. \tag{3.27}$$

The master equation (3.21) is now

$$(S,S) = 2\frac{\partial_r S}{\partial\Phi_i}\frac{\partial_l S}{\partial\Phi^\star_i} = 0, \tag{3.28}$$

and the BRST transformations (3.17)-(3.18) are given by

$$\delta_{BRST}\Phi_i = \frac{\partial_l S}{\partial \Phi_i^\star},\ \ \delta_{BRST}\Phi_i^\star = -\frac{\partial_r S}{\partial \Phi_i}. \tag{3.29}$$

$\Phi_i^\star$ are known as antifields.

Till now we specified the field content of the formalism. The second link to the usual notions of field theory is to demand that "the classical limit" of the odd time Hamiltonian is

$$S(\Phi, \Phi^\star)|_{\Phi^\star=0} = \mathcal{A}[\phi]. \tag{3.30}$$

Let the total number of the phase space variables $(\Phi, \Phi^\star)$ is denoted by $2N$. N of them are "unphysical"(from the odd time formulation point of view) because they are introduced as odd canonical conjugates. On the other hand, by taking the derivative of (3.28) one obtains

$$\left(\frac{\partial_r S}{\partial \Phi_i}\ ,\ \frac{\partial_r S}{\partial \Phi_i^\star} \right) \mathcal{R}_{ij} = 0, \tag{3.31}$$

where

$$\mathcal{R}_{ij} = \begin{pmatrix} \frac{\partial_l \partial_r S}{\partial \Phi_i^\star \partial \Phi_j} & \frac{-\partial_l \partial_r S}{\partial \Phi_i^\star \partial \Phi_j^\star} \\ \frac{\partial_l \partial_r S}{\partial \Phi_i \partial \Phi_j} & \frac{-\partial_l \partial_r S}{\partial \Phi_i \partial \Phi_j^\star} \end{pmatrix}. \tag{3.32}$$

Thus, S is invariant under the gauge transformations generated by $\mathcal{R}$. If $\mathcal{R}$ satisfies

$$\text{rank}\ \mathcal{R}_{ij} = N, \tag{3.33}$$

gauge invariance permits us to eliminate the undesired variables. Moreover $\mathcal{R}$ satisfies

$$\mathcal{R}_{ij}\mathcal{R}_{jk} = 0,$$

so that, N is its maximal rank, which ensures that all of the gauge invariances are taken into account.

Solution of the master equation (3.28) satisfying (3.33) is called proper.

By expressing the “unphysical” variables $\Phi^\star$ in terms of the “physical” ones Φ one can fix the gauge invariance. However, the conditions on their ghost numbers (3.27) do not allow this. Therefore, to achive gauge fixing in this way one should enlarge the space of the original and ghost fields by introducing

$$\Sigma_z,\ \Lambda_z;\ N_{\rm gh}(\Sigma_z) = N_{\rm gh}(\Lambda_z) - 1 = -N_{\rm gh}(\Phi_z),$$

where Φ_z indicate the fileds Φ_i except the original gauge fields. Of course, for not altering the number of the physical variables one should define a new solution of master equation as

$$S_e(\Phi^A, \Phi^\star_A) = S(\Phi_i, \Phi^\star_i) + \Lambda_z \Sigma^\star_z, \tag{3.34}$$

where

$$\Phi^A \equiv (\Phi_i, \Sigma_z, \Lambda_z).$$

Now, gauge fixing can be obtained as

$$\Phi^\star_A = \frac{\partial \Psi(\Phi^A)}{\partial \Phi^A}. \tag{3.35}$$

Obviously, $\Psi(\Phi)$ should be fermionic and moreover, it should possess

$$N_{\rm gh}(\Psi) = -1.$$

Because of these properties Ψ is called “gauge fixing fermion”.

The gauge fixed action

$$S_e(\Phi^A, \partial\Psi/\partial\Phi^A) = S(\Phi_i, \partial\Psi/\partial\Phi_i) + \Lambda_z \frac{\partial\Psi}{\partial\Sigma_z}, \tag{3.36}$$

can be used in the related path integral (partition function)

$$Z = \int \mathcal{D}\Phi^A \exp\{S_e(\Phi^A, \partial\Psi/\partial\Phi^A)\}, \tag{3.37}$$

or in the Green’s function generating functional.

Before proceeding with the general formalism, to illustrate the method let us apply it to Yang-Mills theory.

This theory is described in terms of the action (2.5), which possesses the gauge symmetry given in (2.6). Because of being an irreducible gauge theory, we introduce only the zero stage ghosts η^a, which are anticommuting, and possessing ghost number 1. The minimal set of fields is

$$\Phi_i = (A_\mu^a,\ \eta^a).$$

The odd canonical conjugates (antifields) are

$$\Phi_i^* = (A_\mu^{a*},\ \eta_a^*);\ \epsilon(A^*) = 1,\ \epsilon(\eta^*) = 0;\ N_{\rm gh}(A^\star) = -1, N_{\rm gh}(\eta^\star) = -2.$$

The proper solution of the master equation (3.28) can easily be obtained as

$$S^{YM}(\Phi_i, \Phi_i^\star) = \int d^4x[-\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + A_\mu^{a*}(D^\mu\eta)_a - \frac{1}{2}\eta_a^* f^a_{bc}\eta^b\eta^c]. \tag{3.38}$$

For gauge fixing we enlarge the set of fields by

$$\bar\eta^a, \bar\eta_a^*;\ \ \lambda^a, \lambda_a^*;$$

$$\epsilon(\bar\eta^*) = \epsilon(\bar\eta) + 1 = 0,\ \ \epsilon(\lambda) = \epsilon(\lambda^*) + 1 = 0;$$

$$N_{\rm gh}(\bar\eta) = N_{\rm gh}(\lambda^\star) = -1,\ N_{\rm gh}(\bar\eta^\star) = N_{\rm gh}(\lambda) = 0.$$

The extended proper solution of the master equation (3.34) is

$$S_e^{YM}(\Phi^A, \Phi_A^\star) = S^{YM}(\Phi_i, \Phi_i^\star) - \int d^4x\ \bar\eta_a^*\lambda^a.$$

From this action one can read the BRST transformations by using the definition (3.29), and observe that they are the same with (2.39). We choose the gauge fixing fermion as

$$\Psi = -\bar\eta^a\partial_\mu A_a^\mu,$$

so that the gauge fixed action

$$S_e^{YM}(\Phi^A, \partial\Psi/\partial\Phi^A) = \int d^4x[-\frac{1}{4}F_{\mu\nu}^2 + \lambda\partial^\mu A_\mu + \bar\eta\partial^\mu D_\mu\eta],$$

coincides with (2.38).

B. Quantum Master Equation

The extended solution S_e found by odd time approach is a classical action. The action after taking into consideration quantum corrections will be

$$W(\Phi, \Phi^*) = S_e(\Phi, \Phi^*) + \sum_{n=1}^{\infty} \hbar^n W_n(\Phi, \Phi^*).$$

The gauge fixed action

$$W(\Phi, \frac{\partial \Psi(\Phi)}{\partial \Phi}) \equiv W(\Phi, \Phi^*)|_{\Sigma},$$

can be used in the partition function

$$Z = \int \prod_{x,A} d\Phi^A \exp[-i\hbar W(\Phi, \Phi^\star)|_{\Sigma}]. \tag{3.39}$$

Let the BRST transformation is still given by

$$\delta_{BRST}\Phi^A \equiv (W, \Phi^A)|_{\Sigma} = \frac{\partial W}{\partial \Phi^*_A}|_{\Sigma}. \tag{3.40}$$

Hence the partition function transforms as

$$\delta_{BRST} Z = \int \prod_{x,A} d\Phi^A \left[\frac{\partial_r}{\partial \Phi^A} (\frac{\partial_l W}{\partial \Phi^*_A}|_{\Sigma}) + \frac{i}{\hbar} \frac{\partial_r W|_{\Sigma}}{\partial \Phi^A} \frac{\partial_l W}{\partial \Phi^*_A}|_{\Sigma} \right] \exp[-i\hbar W|_{\Sigma}].$$

Here the former term in the parenthesis is due to the change in the measure. If one demands invariance of the partition function under the BRST transformation (3.40),

$$\frac{-i}{2\hbar}(W, W) + \Delta W + O_1 + O_2 = 0, \tag{3.41}$$

should be satisfied. Here we used the operator

$$\Delta = \frac{\partial_r \partial_l}{\partial \Phi^A \partial \Phi^*_A},$$

and the terms O_1, and O_2 are

$$O_1 = \frac{\partial_r W}{\partial \Phi^*_B} \frac{\partial_r^2 \Psi}{\partial \Phi^A \partial \Phi^B} \frac{\partial_r W}{\partial \Phi^*_A},$$
$$O_2 = \frac{\partial_r \partial_l W}{\partial \Phi^*_B \partial \Phi^*_A} \frac{\partial_r^2 \Psi}{\partial \Phi^A \partial \Phi^B}.$$

By using the symmetry properties one can show that

$$O_1 = -O_1 = 0;\ O_2 = -O_2 = 0.$$

Hence if W satisfies the equation

$$\frac{1}{2}(W, W) + i\hbar\Delta W = 0, \tag{3.42}$$

the partition function (3.39) is invariant under the BRST transformation (3.40). (3.42) is known as the quantum master equation. Now by expanding it in powers of $\hbar$ one obtains

$$(S, S) = \quad 0, \tag{3.43}$$

$$(W_1, S) = \quad i\Delta S, \tag{3.44}$$

$$(W_n, S) = i\Delta W_{n-1} - \frac{1}{2}\sum_{m=1}^{n-1}(W_m, W_{n-m}). \tag{3.45}$$

The terms added after enlarging the minimal set of fields are such that they identically satisfy the master equation and moreover, they do not give any contribution to the quantum corrections of the action. Hence, in (3.43)-(3.45) we can drop them and consider only the minimal set of the fields Φ_i

C. A General Solution of the Master Equation

A proper solution of the master equation can be found by writing S as a polynomial in antifields. This does not cause any difficulty for simple systems like Yang-Mills theory or antisymmetric tensor field. However, usually applying this procedure is complicated for the systems whose level of reducibility is high and/or possess an open gauge algebra. Moreover, a geometrical or algebraic interpretation of the solutions is obscure. Here, we present an easy solution which can be applied to a vast class of gauge theories. Moreover, it is suitable to extract some geometric or algebraic properties of the quantized theory.

One may treat the exterior derivative d and the derivative with respect to the odd time on the same footing by introducing

$$\tilde{d} \equiv d + \partial/\partial\tau. \tag{3.46}$$

In terms of the identification (3.22) $\tilde{d}$ can equivalently be written as

$$\tilde{d} \equiv d + \delta_{BRST}, \tag{3.47}$$

which is defined to satisfy [14], [15]

$$\tilde{d}^2 = d\delta_{BRST} + \delta_{BRST}d = 0.$$

Recall that the exterior derivative d and the BRST transformation δ_{BRST} increase, respectively, differential form degree and ghost number by one.

Consider the minimal set of fields and antifields needed in the BV method of quantization i.e. the original fields, ghosts, ghosts of ghosts, and their antifields. Their main distinguishing parameters are: i)ghost number, ii) behavior under the Lorentz transformations i.e. differential form degree. These two different properties can be unified in terms of the generalized derivative (3.47). i.e. by considering the total degree

$$\mathcal{N} \equiv N_d + N_{\mathrm{gh}}, \tag{3.48}$$

where N_d denotes differential form degree.

A general solution will be given for the gauge systems whose Lagrangian (action) can be put into the first order form

$$L(A,B) = BdA + V(A,B). \tag{3.49}$$

Gauge transformations can be written as

$$\delta^{(0)}(A,B) = R^{(0)}(A,B)\Lambda, \tag{3.50}$$

by suppressing the indices.

The minimal set of fields can be figured out analysing reducibility of the gauge transformations (3.50), as discussed in Section II B. They can be collected in groups as $\tilde{A}$, and $\tilde{B}$ satisfying

$$\mathcal{N}(\tilde{A}) = \mathcal{N}(A)\ ;\ \mathcal{N}(\tilde{B}) = \mathcal{N}(B). \tag{3.51}$$

If differential form degrees of the original fields are various, the above mentioned generalization should be done for each degree. Then, substitute the original fields A, B with the generalized ones $\tilde{A}, \tilde{B}$ in the Lagrangian (3.49):

$$S \equiv L(\tilde{A}, \tilde{B}) = \tilde{B} d\tilde{A} + V(\tilde{A}, \tilde{B}). \tag{3.52}$$

In (3.52) multiplication is defined such that S is a scalar possessing zero ghost number:

$$\mathcal{N}(S) = N_d(S) = N_{\mathrm{gh}}(S) = 0,$$

(3.52) is invariant under the transformations

$$\delta_{\tilde{\Lambda}}(\tilde{A}, \tilde{B}) = \tilde{R}\tilde{\Lambda}, \tag{3.53}$$

where the generators are

$$\tilde{R} \equiv R^{(0)}(\tilde{A}, \tilde{B}), \tag{3.54}$$

and $\tilde{\Lambda}$ is the appropriate generalization of the gauge parameter Λ :

$$\mathcal{N}(\tilde{\Lambda}) = \mathcal{N}(\Lambda).$$

S given by (3.52) is the solution of the master equation if (3.53) can be written as

$$\begin{pmatrix} \delta_{\tilde{\Lambda}} \tilde{A}_i \\ \delta_{\tilde{\Lambda}} \tilde{B}_i \end{pmatrix} = \begin{pmatrix} -\frac{\partial_l \partial_r S}{\partial \tilde{B}_i \partial \tilde{A}_j} & -\frac{\partial_l \partial_r S}{\partial \tilde{B}_i \partial \tilde{B}_j} \\ \frac{\partial_l \partial_r S}{\partial \tilde{A}_i \partial \tilde{A}_j} & \frac{\partial_l \partial_r S}{\partial \tilde{A}_i \partial \tilde{B}_j} \end{pmatrix} \begin{pmatrix} \tilde{\Lambda}_1^j \\ \tilde{\Lambda}_2^j \end{pmatrix}, \tag{3.55}$$

with ($\tilde{\Lambda}_1 \neq 0$, $\tilde{\Lambda}_2 \neq 0$) or ($\tilde{\Lambda}_1 \neq 0$, $\tilde{\Lambda}_2 = 0$) or ($\tilde{\Lambda}_1 = 0$, $\tilde{\Lambda}_2 \neq 0$).

Variation of S under (3.55) can be shown to yield

$$\delta S = \frac{\partial_r (S,S)}{\partial \tilde{A}_j} \tilde{\Lambda}_1^j + \frac{\partial_r (S,S)}{\partial \tilde{B}_j} \tilde{\Lambda}_2^j. \tag{3.56}$$

Obviously, when $\tilde{\Lambda}_1 \neq 0$, and $\tilde{\Lambda}_2 \neq 0$, S satisfies the master equation, and $\tilde{R}$ coincides with (3.32)‡. The same conclusion can be derived when one of the parameters $\tilde{\Lambda}_1$ or $\tilde{\Lambda}_2$

‡$(S,S) = \text{const.} \neq 0$ would lead to the non-consistency of the equations of motion.

is vanishing. Let $\tilde{\Lambda}_1 \neq 0$, $\tilde{\Lambda}_2 = 0$, so that, (S,S) is independent of $\tilde{A}$. (S,S) possesses $N_d = 0$, $N_{\rm gh} = 1$, indicated as $(0,1)$. However, usually it is not possible to construct a function possessing $(0,1)$ degree only in terms of $\tilde{B}$. Hence, we can conclude that (S,S) vanishes. The other case, $\tilde{\Lambda}_1 = 0$, $\tilde{\Lambda}_2 \neq 0$, can be examined similarly.

We choose the signs of the field contents of $\tilde{A}$ and $\tilde{B}$ as

$$\tilde{A}_i = (\Phi_k, \Phi_l^*), \ \tilde{B}_i = (-\Phi_k^*, \Phi_l),$$

so that, the transformations

$$\delta \tilde{A}_i = \frac{\partial_r S}{\partial \tilde{B}_i}, \ \delta \tilde{B}_i = -\frac{\partial_r S}{\partial \tilde{A}_i}, \tag{3.57}$$

define the BRST transformations in accordance with the BV formalism (3.29). Obviously, in (3.57) the right hand side is defined to have one more ghost number, but the same N_d of the field appearing on the left hand side.

By construction $S(\tilde{A}, \tilde{B})$ possesses the correct classical limit:

$$S|_{\Phi^\star=0} = L(A,B).$$

Moreover, in $\tilde{A}$ and $\tilde{B}$ all the fields of the minimal sector are included, and because of the form of S, (3.52),

$$\mathrm{rank} \left| \frac{\partial^2 S}{\partial(\tilde{A},\tilde{B})\partial(\tilde{A},\tilde{B})} \right| = N,$$

where N is the number of the components of $\tilde{A}$ or $\tilde{B}$. This is the condition given in (3.33). Hence, we conclude that under the above mentioned conditions $S = L(\tilde{A}, \tilde{B})$ is the proper solution of the master equation.

D. Examples to the General Solution

1. Yang-Mills Theory

The first order action

$$L = \frac{-1}{2}\int d^4x\ (B_{\mu\nu}F^{\mu\nu} - \frac{1}{2}B_{\mu\nu}B^{\mu\nu}), \tag{3.58}$$

is equivalent to (2.5) on mass shell, and it is invariant under the infinitesimal gauge transformations

$$\delta A_\mu = D_\mu \Lambda\ ,\ \ \delta B_{\mu\nu} = [B_{\mu\nu}, \Lambda].$$

The theory is irreducible, so that for the covariant quantization we need to introduce (in the minimal sector) the ghost field $\eta_{(0,1)}$, and the antifields $A^\star_{(3,-1)}$, $\eta^\star_{(4,-2)}$, and $B^\star_{(2,-1)}$. The first number in parenthesis is the differential form degree and the second is the ghost number. Here the star indicates the antifields as well as the Hodge-map.

By using (3.51) we write the generalized fields as

$$\begin{aligned}\tilde{A} &= A_{(1,0)} + \eta_{(0,1)} + B^\star_{(2,-1)},\\ \tilde{B} &= -A^\star_{(3,-1)} - \eta^\star_{(4,-2)} + B_{(2,0)}.\end{aligned}$$

In terms of the substitution

$$A \rightarrow \tilde{A},\ \ B \rightarrow \tilde{B},$$

in (3.58) one obtains

$$S = \frac{-1}{2}\int d^4x\ [\tilde{B}(d\tilde{A} + \tilde{A}\tilde{A}) - \frac{1}{2}\tilde{B}\tilde{B}]. \tag{3.59}$$

By using the property of the multiplication that the scalar product is different from zero only when its ghost number vanishes, we get

$$S = -\int d^4x\ (\frac{1}{2}B_{\mu\nu}F^{\mu\nu} - B^{\mu\nu}[\eta, B^\star_{\mu\nu}] + A^\star_\mu D^\mu\eta + \frac{1}{2}\eta^\star[\eta,\eta] - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}). \tag{3.60}$$

We may perform a partial gauge fixing $B^\star = 0$, and then use the equations of motion with respect to $B_{\mu\nu}$ to obtain

$$S \rightarrow S^{YM} = \int d^4x\ (-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A^\star_\mu D^\mu\eta - \frac{1}{2}\eta^\star[\eta,\eta]),$$

which is the minimal solution of the master equation for Yang-Mills theory (3.38). One can also observe that (3.60) is the desired solution by denoting that indeed, the transformations (3.57) are the BRST transformations: using (3.59) in (3.57) leads to

$$\delta\tilde{A} = \tilde{F} - \tilde{B},\ \delta\tilde{B} = -\tilde{D}\tilde{B}, \tag{3.61}$$

where $\tilde{D} = d + [\tilde{A},]$ and $\tilde{F}$ is the related curvature. Formally we have

$$\begin{aligned}
\delta^2\tilde{A} &= \tilde{D}\cdot(\tilde{F} - \tilde{B}) + \tilde{D}\tilde{B} = 0,\\
\delta^2\tilde{B} &= -\tilde{D}\cdot\tilde{D}\tilde{B} + (\tilde{F} - \tilde{B})\cdot\tilde{B} = 0,
\end{aligned}$$

due to the Bianchi identities $\tilde{D}\cdot\tilde{F} = 0$, the definition of the curvature $\tilde{F} = \tilde{D}\cdot\tilde{D}$, and $\tilde{B}\cdot\tilde{B} = \tilde{B}_i\tilde{B}_j - (-1)^{\epsilon(\tilde{B}_i)\epsilon(\tilde{B}_j)}\tilde{B}_j\tilde{B}_i = 0$.

In the gauge $B^\star = 0$ use of the equations of motion $B_{\mu\nu} = F_{\mu\nu}$ in (3.61) yields

$$(\delta + d)(A+\eta) + [(A+\eta),(A+\eta)] = F$$

which is the Maurer-Cartan horizontality condition [14].

2. The Self-interacting Antisymmetric Tensor Field

As we have seen, (2.8)-(2.11), this system is a first stage reducible theory. Hence we need to introduce the noncommuting ghost and commuting ghost of ghost fields

$$C_0^\mu,\ C_1;\ \mathrm{N_{gh}}(C_0^\mu) = 1,\ \mathrm{N_{gh}}(C_1) = 2.$$

After introducing the related antifields, the generalized fields can be written as

$$\begin{aligned}
\tilde{A} = &\ A_{(1,0)} + B^\star_{(2,-1)} + C^\star_{0(3,-2)} + C^\star_{1(4,-3)},\\
\tilde{B} = &\ -A^\star_{(3,-1)} + B_{(2,0)} + C_{0(1,1)} + C_{1(0,2)}.
\end{aligned}$$

By following the general procedure we find

$$S = -\int d^4x\ [\tilde{B}(d\tilde{A} + \frac{1}{2}\tilde{A}\tilde{A}) - \frac{1}{2}\tilde{A}\tilde{A}], \tag{3.62}$$

which yields

$$S=-\int d^4x\ \{B_{\mu\nu}F^{\mu\nu}+2\epsilon_{\mu\nu\rho\sigma}C_0^{\mu}D^{\nu}B^{\star\rho\sigma}+2C_1D^{\mu}C^{\star}_{0\mu}$$
$$+\epsilon^{\mu\nu\rho\sigma}C_1[B^{\star}_{\mu\nu},B^{\star}_{\rho\sigma}]-\frac{1}{2}A_{\mu}A^{\mu}\}, \tag{3.63}$$

in terms of the components. This is the minimal solution of the master equation of the theory defined by (2.8) [16], which can also be deduced by observing that the transformations

$$\delta\tilde{A}=\tilde{F},\ \ \delta\tilde{B}=-\tilde{D}\tilde{B}+\tilde{A}, \tag{3.64}$$

found by substituting (3.62) into (3.57), satisfy

$$\begin{aligned}\delta^2\tilde{A}\ &=\ \tilde{D}\cdot\tilde{F}=0,\\ \delta^2\tilde{B}\ &=\ -\tilde{D}\cdot(-\tilde{D}\tilde{B}+\tilde{A})-\tilde{F}\tilde{B}+\tilde{F}=0,\end{aligned}$$

due to the Bianchi identities and the definition of the curvature.

3. Chern-Simons theory in $d=2p+1$

By examining the reducibility properties (2.18) of the theory given by (2.12)-(2.14), one introduces ghosts, ghosts of ghosts and the related antifields which lead to the generalized fields

$$\begin{aligned}\tilde{\phi}&=\sum_{i=0}^{p-1}\left[\phi_{(2i+1,0)}+\sum_{j=1}^{2i+1}\eta_{(2i+1-j,j)}+\phi^{*}_{(2i+2,-1)}+\sum_{j=-2p+2i+4}^{-2}\eta^{*}_{(2i+1-j,j)}\right]\\ \tilde{\psi}&=\sum_{i=0}^{p}\psi_{(2i,0)}+\sum_{i=1}^{p}\sum_{j=1}^{2i}\kappa_{(2i-j,j)}+\sum_{i=0}^{p}\psi^{*}_{(2i+1,-1)}+\sum_{i=0}^{p-1}\sum_{j=-2p+2i+1}^{-2}\kappa^{*}_{(2i-j,j)}.\end{aligned} \tag{3.65}$$

The antifield of the field $a_{(k,l)}$ is defined as $a^{*}_{(2p+1-k,-l-1)}$. Observe that $\tilde{\phi}$ and $\tilde{\psi}$ are, respectively, collection of $2i+1$-forms and $2i$-forms. Now, in terms of $\tilde{A}=\tilde{\phi}+\tilde{\psi}$, we can write

$$S_d=\frac{1}{2}\int_{M_d}\left(\tilde{A}d\tilde{A}+\frac{2}{3}\tilde{A}^3\right). \tag{3.66}$$

In terms of $\tilde{\phi}$ and $\tilde{\psi}$ components (3.66) yields

$$S_d=\int_{M_d}\left(\tilde{\phi}d\tilde{\phi}+\frac{1}{3}\tilde{\phi}^3+\tilde{\psi}(d+\tilde{\phi})\tilde{\psi}\right). \tag{3.67}$$

S_d is the proper solution of the master equation, because it is invariant under the transformations generated by

$$\tilde{R} = d + [\tilde{A},] = \frac{\partial^2 S_d}{\partial \tilde{A}^2},$$

due to the generalization of (2.15). Because of the sign assignments in (3.65) the transformations (3.55) are given as

$$\delta_{\tilde{\Sigma}} \tilde{A}_i = \omega_{ij} \frac{\partial_l \partial_r S}{\partial \tilde{A}_j \partial \tilde{A}_k} \tilde{\Sigma}_k; \ \omega_{ij} = \left(\begin{array}{cc} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{array} \right),$$

where the generalized gauge parameter is

$$\begin{aligned}
&\tilde{\Sigma} = \tilde{\Lambda} + \tilde{\Xi}, \\
&\tilde{\Lambda} = \sum_{i=0}^{p-1} \Lambda_{(2i,0)} + \sum_{i=1}^{p-1} \sum_{j=1}^{2i} \lambda_{(2i-j,j)} + \sum_{i=0}^{p-1} \Lambda^*_{(2i+1,-1)} + \sum_{i=0}^{p-2} \sum_{j=-2p+2i+1}^{-2} \lambda^*_{(2i-j,j)} \\
&\tilde{\Xi} = \sum_{i=0}^{p-1} \left[\Xi_{(2i+1,0)} + \sum_{j=1}^{2i+1} \xi_{(2i+1-j,j)} + \Xi^*_{(2i+2,-1)} + \sum_{j=-2p+2i+4}^{-2} \xi^*_{(2i+1-j,j)} \right].
\end{aligned}$$

Observe that the transformations given by (3.57) by making use of (3.66), leads to

$$\delta \tilde{A} = \tilde{F},$$

so that

$$\delta^2 \tilde{A} = 0,$$

following from the Bianchi identities.

This example is somehow different from the general case, because the total degree of the components of A are not the same. But the integral selects only the terms with the correct degree. One could write the solution of the master equation by using the generalized forms each of which possessing only one degree, and then gather them to obtain (3.66).

Although the gauge generators satisfy an open algebra, the gauge theory of the quadratic Lie algebra (2.22)-(2.26), is an irreducible theory. Thus, one needs to introduce only one family of ghosts η^a. The generalized fields are

$$\tilde{h} = h_{(1,0)} + \eta_{(0,1)} + \Phi^{\star}_{(2,-1)}, \tag{3.68}$$

$$\tilde{\Phi} = -h^{\star}_{(1,-1)} - \eta^{\star}_{(2,-2)} + \Phi_{(0,0)}. \tag{3.69}$$

Now, by replacing the fields $h,\ \Phi$ with the generalized ones $\tilde{h},\ \tilde{\Phi}$ in (2.22) one obtains

$$S = -\int d^2x \frac{1}{2}\{\tilde{\Phi}_a(d\tilde{h}^a + f_{bc}{}^a\tilde{h}^b\tilde{h}^c + V^{ad}_{bc}\tilde{\Phi}_d\tilde{h}^b\tilde{h}^c) + k_{ab}\tilde{h}^a\tilde{h}^b\}. \tag{3.70}$$

It is the solution of the master equation, because S is invariant under the gauge transformation obtained as the generalization of the original ones (2.25)-(2.26), which can be written as in (3.55):

$$\begin{pmatrix} \delta_{\tilde{\lambda}}\tilde{h} \\ \delta_{\tilde{\lambda}}\tilde{\Phi} \end{pmatrix} = \begin{pmatrix} d + f\tilde{h} + 2V\tilde{\Phi}\tilde{h} & V\tilde{h}\tilde{h} \\ f\tilde{\Phi} + V\tilde{\Phi} + k & d + f\tilde{h} + 2V\tilde{\Phi}\tilde{h} \end{pmatrix} \begin{pmatrix} \tilde{\lambda} \\ 0 \end{pmatrix}.$$

As we discussed in Section III C, this symmetry yields

$$\frac{\partial(S,S)}{\partial\tilde{h}} = 0,$$

but (S,S) cannot depend only on $\tilde{\Phi}$ because there is not any field in $\tilde{\Phi}$ whose total degree is $(0,1)$, so that

$$(S,S) = 0.$$

(3.70) in components yields

$$\begin{aligned} S = \quad & \int d^2x\{\mathcal{L} + h^{\star\mu}_a(\partial_\mu\eta^a + f_{ba}{}^c\Phi_c\eta^b + 2V^{ad}_{bc}\Phi_d h^b_\mu\eta^c) \\ & + \Phi^{\star a}(f_{ba}{}^c\Phi_c\eta^b + V^{cd}_{ba}\Phi_c\Phi_d\eta^b + k_{ba}\eta^b) \\ & + \eta^{\star}_a(\tfrac{1}{2}f_{bc}{}^a\eta^b\eta^c + V^{ad}_{bc}\Phi_d\eta^b\eta^c) - \tfrac{1}{2}\epsilon_{\mu\nu}V^{ad}_{bc}h^{\star\mu}_a h^{\star\nu}_d\eta^b\eta^c\}. \end{aligned} \tag{3.71}$$

Indeed, (3.71) is the proper solution of the master equation as one can check explicitly.

E. Discussions

One can apply the formalism used in this paper to other gauge systems like topological quantum field theories [15], [19] (for a review of topological field theories see [17]) and covariant string field theories. In fact, without realizing the general formalism, it was shown in the quantization of the Neveu-West covariant string field theory [18] that the generalized fields of Section III C can be used to write the proper solution of the master equation [20].

Although, gauge fixing can be performed in a compact way in terms of generalized fields, particular properties of the gauge system considered are essential to discuss it in a concrete (not formal) manner. Because of not being involved with a specific gauge theory, here we discussed the gauge fixing on general grounds without applying it to the examples considered.

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